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On a Class of Singular Perturbation Problems

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I. INTRODUCTION

The study of certain boundary value problems in the theory of thin elastic plates and shells leads to a class of singular perturbation problems with some interesting features.

For example the differential equation

$$\Delta\Delta u - \lambda u_{yy} = 0 \tag{1.1}$$

occurs in the theory of small transverse deflections of stretched thin elastic plates. Here $u(x, y)$ is the deflection of the midplane of the plate, $\Delta u \equiv u_{xx} + u_{yy}$, and differentiation is indicated by subscripts. In the situation of interest, λ is assumed large; it depends on the tension in the plate, the plate thickness, and certain material constants.

A second example concerns the system of differential equations for the stress function $\varphi(x, y)$ and the transverse deflection $w(x, y)$ of the shallow hyperbolic paraboloidal shell. This system can be reduced to the form

$$\begin{aligned} \Delta\Delta w + \lambda^3 \varphi_{xy} &= 0, \\ \Delta\Delta \varphi - \lambda^3 w_{xy} &= 0, \end{aligned} \tag{1.2}$$

where the parameter λ depends on the thickness, geometry and material properties of the shell.

In both problems the domain R of the independent variables which is of particular interest is the rectangle $0 < x < \gamma$, $0 < y < 1$, and there are suitable subsidiary conditions imposed on the boundary S of R. In both of these problems one is interested in obtaining asymptotic approximations to the solutions for large λ.

The "reduced equations" which result from (1.1) and (1.2) when only the terms involving λ are retained are

$$u_{yy} = 0, \tag{1.3}$$

in the case of (1.1), and

$$\begin{aligned} w_{xy} &= 0, \\ \varphi_{xy} &= 0, \end{aligned} \tag{1.4}$$

for (1.2). As is common in singular perturbation problems, these reduced equations are of lower order than the entire equations from which they are derived. More important from our viewpoint, however, is the fact that the reduced equations are of parabolic and hyperbolic types, while the original equations are elliptic. *Furthermore, the boundary S of the domain coincides either partially or entirely with portions of characteristic curves of the reduced equations.* In the case of (1.1), the segments $x = 0$ and $x = \gamma$, for $0 \leqslant y \leqslant 1$, coincide with portions of the characteristics $x = 0$ and $x = \gamma$ of the parabolic equation (1.3), while the segments $y = 0$ and $y = 1$, for $0 \leqslant x \leqslant \gamma$, are noncharacteristic. In the case of (1.2), all four sides of the rectangular boundary coincide with portions of the characteristic curves $x =$ constant and $y =$ constant of the pair of hyperbolic equations (1.4).

In this paper we wish to indicate by means of a simple example the special nature of the asymptotic approximation process which is encountered when a portion of the boundary of the domain coincides with a characteristic of the reduced equation. Although this example is much simpler in detail than the elasticity problems mentioned above, it nevertheless preserves some (but not all) of their interesting features. Moreover the exact solution is available to confirm and make precise results obtained by a formal "boundary layer" technique.

Certain aspects of the role of the characteristics of the reduced equation in problems of boundary layer type have been discussed by Latta [1]. In connection with a problem concerning flow of a viscous fluid past a flat plate, he pointed out the exceptional character of the boundary layer along a portion of the boundary which coincides with a characteristic of the reduced equation.

The situation of a "characteristic boundary" has also been discussed by Višik and Lyusternik in their extensive study of problems of boundary layer type [2].[1] They have shown that it gives rise to the so-called "parabolic

[1] The authors are indebted to Professor A. Erdélyi of the California Institute of Technology for calling to their attention the translation [2] of the work of Višik and Lyusternik.

boundary layer" whose determination depends on the integration of a parabolic partial differential equation. Questions which arise in connection with the determination of suitable boundary conditions for this boundary layer differential equation, and which become particularly crucial for higher order problems, are not discussed in [2]. It is primarily to these questions that we shall give our attention here, and in which the corner layer plays an important role.

A reference to the occurrence of a parabolic partial differential equation in the description of a boundary layer along a characteristic also occurs in [3].

The techniques which arise in the context of the simple example to be discussed here, form, after suitable elaboration, the basis of a systematic approximation procedure for treating certain elasticity problems. This is particularly true of what we shall subsequently call "corner layers." Although the example to be treated in this paper serves to illustrate the corner layer notion, its essential usefulness is more fully exhibited in higher order problems, such as certain boundary value problems connected with (1.1) and (1.2)

A detailed discussion of plate and shell problems of the type mentioned above, as well as problems of a similar nature for thin helicoidal shells which have been considered by one of us,[2] will be reported elsewhere. We confine ourselves here to a discussion of the example to be formulated in the following section.

II. The Problem

It is required to find the behavior for large λ of the function $u(x, y; \lambda)$ satisfying the differential equation

$$\Delta u - \lambda u_y = 0 \tag{2.1}$$

in the semi-infinite strip R: $0 < x < \infty$, $0 < y < 1$, and the boundary conditions

$$u(0, y; \lambda) = f(y), \qquad 0 \leqslant y \leqslant 1. \tag{2.2a}$$

$$u(x, 0; \lambda) = 0, \qquad 0 < x < \infty, \tag{2.2b}$$

$$u(x, 1; \lambda) = 0, \qquad 0 < y < \infty. \tag{2.2c}$$

The function u and its partial derivatives are required to be bounded as $x \to \infty$. It will be assumed that f possesses a continuous derivative on $0 \leqslant y \leqslant 1$.

[2] R.E.M.

The characteristics of the reduced equation

$$u_y = 0 \tag{2.3}$$

are the lines $x =$ constant. The end segment $x = 0$ of the boundary of R thus coincides with a characteristic of (2.3).

It is of course possible to construct the exact solution to (2.1) satisfying (2.2) by entirely elementary methods. It can be used to show the precise sense in which the approximations (obtained in a heuristic way in the following section) are valid.

The problem of the behavior of the solutions of (2.1) for large λ has been discussed by Wasow [4] for more general (bounded) domains. He does not treat, however, the "boundary layer" part of the solution.

Levinson [5] treats the first boundary value problem for an equation which includes

$$\lambda^{-1}\Delta u - (Au_x + Bu_y + Cu) = 0 \tag{2.4}$$

where λ^{-1} is small, A, B, and C are functions of x and y, and the domain is again quite general. While the conclusions of the theorem in [5] are relevant to certain portions of the domain R in our problem, they cannot be used to find the boundary layer behavior of the solution along the characteristic segment at $x = 0$.

In [6] Kamenomostskaya treats the problem considered by Levinson but in the case of a multiply connected domain whose boundary is a union of *closed* characteristic curves of the reduced equation associated with (2.4).

A boundary value problem very similar to the one to be treated in detail here is discussed by Višik and Lyusternik in [2, pp. 301-304]. They consider the differential equation

$$\epsilon^2\Delta u + u_x - u = h, \tag{2.5}$$

where h is a given function of x and y, and the domain of interest is the rectangle $0 < x < a$, $0 < y < b$. The solution u is required to vanish on the boundary of the rectangle. Here, of course, the parameter ϵ is small, and the reduced equation obtained by setting $\epsilon = 0$ has the lines $y =$ constant as its characteristics. This problem possesses essentially the same qualitative features as the one to be investigated here. However, the treatment in the present paper brings out several aspects of the problem which are not discussed in [2].

There is a simple physical situation which gives rise to the boundary value problem (2.1) and (2.2), and which provides a useful interpretation of the boundary layer behavior which we obtain later. Consider a thin semi-infinite heat conducting plate P moving between two heat reservoirs A and B with a constant velocity v in the positive y direction (see Fig. 1). The portion of the plate which is exposed at any instant of time occupies the semi-infinite strip R: $0 < x < \infty$, $0 < y < 1$. The reservoirs A and B are maintained at temperature zero, and a fixed temperature distribution $u = f(y)$ is placed on the exposed edge $x = 0$, $0 < y < 1$. Heat losses through the exposed lateral surfaces of the plate are assumed negligible. The thinness of the plate is assumed to make variations in temperature through the thickness negligible and to allow the assumption that the portions of the plate in contact with a reservoir have the temperature of that reservoir. If u denotes the temperature distribution in the exposed region of the plate, and if this temperature distribution is steady so that $\partial u/\partial t = 0$, then the determination of u can be reduced to the problem (2.1), (2.2). The parameter λ occurring in (2.1) is proportional to the velocity v of the moving plate. Since λ is large, convection must play a dominant role; the diffusion mechanism operates to smooth any discontinuities introduced by the convection.

A nonhomogeneous version of the differential equation (2.1) occurs in one of the simpler models of the problem of wind-driven ocean circulation (see [7, p. 54]).

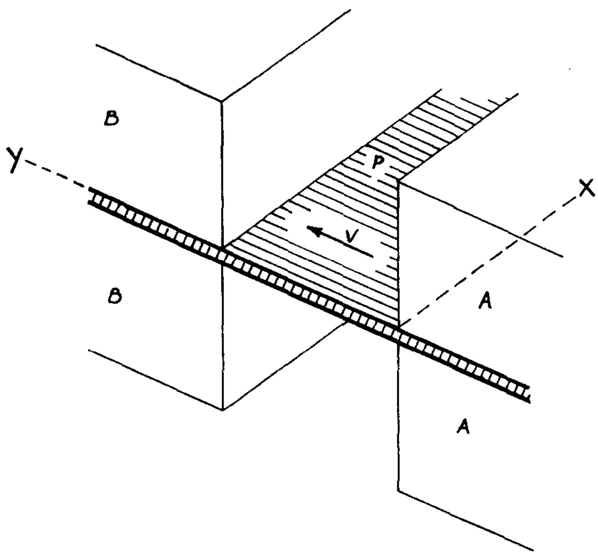

FIG. 1. Plate P moving between heat reservoirs A and B

III. Interior, Edge, and Corner Approximations

Standard asymptotic procedures suggest that, in the interior of R, u will be approximated by

$$u(x, y, \lambda) \sim u_0(x) \tag{3.1}$$

for some function $u_0(x)$, in view of the reduced equation (2.3). In the present case it is possible to satisfy the boundary conditions at $y = 0$ and $y = 1$ by taking $u_0(x) \equiv 0$.[3] Thus it is to be expected that

$$u(x, y, \lambda) \sim 0 \tag{3.1 a}$$

will be a good approximation except near the end $x = 0$, where the boundary condition (2.2a) is violated.

To obtain the required "boundary layer" correction near $x = 0$, the approximation (3.1) is replaced by

$$u(x, y; \lambda) \sim u_0(x) + u_1(\xi, y) = u_1(\xi, y), \tag{3.2}$$

where

$$\xi = \lambda^{1/2} x \tag{3.3}$$

is a boundary layer variable. If the original differential equation is transformed by introducing (3.3) into (2.1), a *new* reduced equation is obtained when only the terms proportional to λ are retained. The function $u_1(\xi, y)$ is required to satisfy this reduced differential equation.

$$\frac{\partial^2 u_1}{\partial \xi^2} - \frac{\partial u_1}{\partial y} = 0, \qquad 0 < \xi < \infty, \qquad 0 < y < 1. \tag{3.4}$$

If u_1 is to represent a boundary layer correction, it (and its derivatives) must be bounded by a decaying exponential in ξ for $\xi > 0$.

Since $u_1(\xi, y)$ satisfies the partial differential equation (3.4) in the domain $0 < \xi$, $0 < y < 1$ the requirements that u_1 decay exponentially in ξ and that

$$u_1(0, y) = f(y), \qquad 0 < y < 1, \tag{3.5}$$

are not sufficient to determine u_1 uniquely; an additional boundary condition is required. In the present example, involving as it does the well known heat equation,[4] it is possible to convince oneself by direct (but *ad hoc*) arguments that the additional boundary condition for (3.4) should be the vanishing of u_1 at $y = 0$ (Eq. (2.2b)), rather than at $y = 1$. However, in the more complicated elasticity problems mentioned in the Introduction, a systematic procedure is needed for deducing additional boundary conditions for the boundary layer differential equations. The argument to be described below for the present problem can, with suitable elaborations, be used in some of these higher order problems.

[3] For more general boundary conditions along $y = 0$, $y = 1$, see Section V.

[4] A boundary layer term satisfying the heat conduction equation (3.4) also occurs in a problem discussed in [8].

The need for an additional boundary condition when the boundary layer is determined by a partial differential equation arises in an entirely similar fashion in the problem treated in [2] in connection with the differential equation (2.5). However no rationale is given in [2] for determining the appropriate additional condition; it is stated and used, but not derived.

Since u_1 cannot satisfy both conditions (2.2a) and (2.2b), it must fail to approximate u near one (or conceivably both) of the two corners $x = 0$, $y = 0$ or $x = 0$, $y = 1$. Suppose $u_1(\xi, y)$ fails to vanish at $y = 1$. Then we introduce a "corner layer" correction u_2 by replacing the approximation (3.2) by

$$u(x, y; \lambda) \sim u_1(\xi, y) + u_2(\xi, \eta), \tag{3.6}$$

where

$$\eta = \lambda(1 - y) \tag{3.7}$$

is a boundary layer variable at the edge $y = 1$. We emphasize that ξ is the same variable (3.3) which appears in the boundary layer $u_1(\xi, y)$. *This means that we have chosen the "x-scales" of u_1 and u_2 to be the same, so that u_2 will match u_1 to the boundary condition $u = 0$ at $y = 1$.* In order to have the correction $u_2(\xi, \eta)$ confined to the corner $x = 0$, $y = 1$, u_2 must have the exponential decay property in both ξ and η. Using this property with (3.6) and (3.7), it is easily found that u_2 satisfies the differential equation

$$\frac{\partial^2 u_2}{\partial \eta^2} + \frac{\partial u_2}{\partial \eta} = 0, \qquad 0 < \eta < \infty. \tag{3.8}$$

The boundary condition (2.2c) then gives, from (3.6),

$$u_1(\xi, 1) + u_2(\xi, 0) = 0, \qquad 0 < \xi < \infty. \tag{3.9}$$

A similar discussion may be given for an exponentially decaying corner layer correction u_3 near $x = 0$, $y = 0$. Equation (3.6) is replaced by

$$u(x, y; \lambda) \sim u_1(\xi, y) + u_2(\xi, \eta) + u_3(\xi, \bar{\eta}) \tag{3.10}$$

where

$$\bar{\eta} = \lambda y \tag{3.11}$$

is the boundary layer variable referring to the edge $y = 0$. Corresponding to (3.8) and (3.9) there now follows

$$\frac{\partial^2 u_3}{\partial \bar{\eta}^2} - \frac{\partial u_3}{\partial \bar{\eta}} = 0 \tag{3.12}$$

and

$$u_1(\xi, 0) + u_3(\xi, 0) = 0. \tag{3.13}$$

The upper corner layer correction u_2 is of course absent from (3.13) as it must be exponentially small in the lower corner.

Solving the problem represented by (3.8) and (3.9) gives the corner layer correction u_2 near $x = 0, y = 1$ as

$$u_2(\xi, \eta) = - u_1(\xi, 1)\, e^{-\eta}. \tag{3.14}$$

Next from (3.12) and (3.13) the lower corner layer is

$$u_3(\xi, \bar{\eta}) = - u_1(\xi, 0)\, e^{\bar{\eta}}. \tag{3.15}$$

But the exponential decay requirement for the lower corner layer obviously fails to hold unless

$$u_1(\xi, 0) = 0, \qquad \xi > 0. \tag{3.16}$$

This is the appropriate additional boundary condition required for (3.4). Moreover it follows that the corner layer u_3 is not present in the problem. The impossibility of a lower corner layer u_3 has thus led us to require that u_1 vanish at the lower corner. The error introduced in this way at the *upper corner* will be removed by the upper corner layer u_2.

The boundary layer correction $u_1(\xi, y)$ now must satisfy the differential equation (3.4), the boundary condition (3.5) and the "initial" condition (3.16). If the differential equation (3.4) were required to hold in the entire quadrant $\xi > 0$, $y > 0$, instead of the strip $\xi > 0$, $0 < y < 1$, and if the boundary value $f(y)$ were given along the entire half-line $\xi = 0, y \geqslant 0$, then the boundary-initial value problem would be a standard one in heat conduction theory whose solution is

$$u_1(\xi, y) = (2/\pi)^{1/2} \int_{(2y)^{-1/2}\xi}^{\infty} e^{-t^2/2} f(y - \xi^2/2t^2)\,dt. \tag{3.17}$$

Since the values of this solution, for $0 < y < 1$, depend only on the values of $f(y)$ for $0 < y < 1$, we expect that our problem for u_1 is well posed and that its solution is given by (3.17).

Let $M = \max |f(y)|$, $0 \leqslant y \leqslant 1$. Then clearly

$$|u_1(\xi, y)| \leqslant \frac{M}{\sqrt{2}} \operatorname{erfc}\left(\frac{\xi}{(2y)^{1/2}}\right).$$

From known properties of the complementary error function, it follows that u_1 decays like $(y/\pi\xi^2)^{1/2} \exp(-\xi^2/2y)$ as ξ increases. We may thus think of the boundary layer correction u_1 as important in the thin layer near $x = 0$ bounded by the parabola $\xi/(2y)^{1/2} = 1$. A boundary layer of the type given in (3.17) has been called a "parabolic boundary layer" by Višik and Lyusternik [2].

From (3.14) and (3.17) the upper corner layer is found to be

$$u_3(\xi, \eta) = -\left(\frac{2}{\pi}\right)^{1/2} e^{-\eta} \int_{\xi/\sqrt{2}}^{\infty} f\left(1 - \frac{\xi^2}{2t^2}\right) e^{-t^2}\, dt. \tag{3.18}$$

The approximation to the solution at the present stage is

$$u(x, y; \lambda) \sim u_1(\lambda^{1/2}x, y) + u_2[\lambda^{1/2}x, \lambda(1 - y)]. \tag{3.19}$$

From this it is clear that the upper corner layer u_2 may reintroduce an error in the satisfaction of the boundary conditions at $x = 0$. This error is exponentially small *except* within a distance of order λ^{-1} from the corner $x = 0$, $y = 1$. To correct this, another corner layer $u_4(\zeta, \eta)$ must be introduced, with $\zeta = \lambda x$ and η still given by (3.7). Thus u_4 may be matched with u_2 in order to restore the boundary condition at $x = 0$. It turns out that u_4 must satisfy the *full* original differential equation (2.1) in the form

$$\frac{\partial^2 u_4}{\partial \zeta^2} + \frac{\partial^2 u_4}{\partial \eta^2} - \frac{\partial u_4}{\partial \eta} = 0, \qquad 0 < \zeta, \eta < \infty, \tag{3.20}$$

and the boundary conditions

$$u_4(\zeta, 0) = 0, \qquad 0 < \zeta < \infty \tag{3.21a}$$

$$u_4(0, \eta) = -\lim_{\xi \to 0} u_2(\xi, \eta) = \lim_{\xi \to 0} u_1(\xi, 1)\, e^{-\eta} = f(1)\, e^{-\eta}. \tag{3.21b}$$

The function u_4 must of course decay exponentially in ζ and η. Since the differential equation for u_4 is the full original equation (2.1), and suitable boundary conditions can be satisfied at $x = 0$ and at $y = 1$, it is to be expected that no further matching is required in the upper corner. Note that, according to (3.21b), this "ultimate" corner layer u_4 will be absent if $f(1) = 0$. The function u_4 can be computed explicitly, but we shall not give it here.

A lower corner layer u_5 analogous to u_4 would not be expected to be present, since no significant error has been introduced in the lower corner. While we will not discuss higher order approximations in this paper, we remark that there is reason to believe that certain unusual features arise when higher order terms are investigated. This would seem to be the case particularly in the lower corner. It may be, for example, that while u_5 is not present to first in λ^{-1}, it is present in the higher order terms.

IV. Results of a Comparison with the Exact Solution

The explicit solution to the original boundary value problem (2.1) and (2.2) can be obtained readily in several forms. Using a representation which involves the expansion of the Green's function in a series of modified Bessel functions, it is possible to prove the statements which we list below. Since the estimates involved are tedious and pertain only to this special problem, we omit the proofs.

We recall that $f(y)$, continuously differentiable for $0 \leqslant y \leqslant 1$, is the boundary value of u at $x = 0$. Throughout the remainder of this section, δ denotes an arbitrary (small) positive number, R is the interior of the semi-infinite strip $x > 0$, $0 < y < 1$, and S is its boundary. The symbols D^δ and D_δ represent quarter-discs at (0, 1) and (0, 0), respectively, and are defined as follows.

$$D^\delta = \{(x, y) \mid x \geqslant 0, y \leqslant 1, x^2 + (y-1)^2 < \delta^2\},$$

$$D_\delta = \{(x, y) \mid x \geqslant 0, y \geqslant 0, x^2 + y^2 < \delta^2\}.$$

We list here the results which follow from an examination of the exact solution.

(i) There exist positive constants $a(\delta)$, $M(\delta)$ such that

$$| u(x, y; \lambda) | < Me^{-a\lambda}$$

for all $x \geqslant \delta$, $0 \leqslant y \leqslant 1$. Thus the interior solution is zero to all orders in λ^{-1}.

(ii) There exists a positive constant $M(\delta)$ such that

$$| u(x, y; \lambda) - u_1(\lambda^{1/2}x, y) | < M\lambda^{-1} \tag{4.1}$$

for $(x, y) \in R + S - D^\delta - D_\delta$. The boundary layer u_1 is a uniformly valid asymptotic approximation to u to order λ^{-1} on any closed subdomain of $R + S$ which excludes the corners (0, 0) and (0, 1).

(iii) If $f(y) = O(y)$ as $y \to 0+$, then there exists a constant $M(\delta) > 0$ such that (4.1) holds for

$$(x, y) \in R + S - D^\delta .$$

The boundary layer is a uniformly valid asymptotic approximation to order λ^{-1} *everywhere* outside the upper corner if $f(y)$ vanishes like y at $y = 0$.

(iv) If $f(y) = O(y)$ as $y \to 0+$ and $f(y) = O(1-y)$ as $y \to 1-$, then there exists a positive constant M such that

$$| u(x, y; \lambda) - u_1(\lambda^{1/2}x, y) - u_2[\lambda^{1/2}x, \lambda(1-y)] | \leqslant M\lambda^{-1}$$

for all $(x, y) \in R + S$.

Therefore, the boundary layer u_1 and the upper corner layer u_2 together provide a uniformly valid asymptotic approximation to order λ^{-1} to u on the closed domain $R + S$ when f vanishes suitably at $y = 0$ and $y = 1$.

In the boundary value problem for the differential equation (2.5) on a rectangle as discussed in [2], only the interior approximation and the boundary layers are obtained; the notion of the corner layer does not appear. Thus the asymptotic approximation obtained is not valid on the closed rectangle, but only in the set obtained by excluding neighborhoods of certain corners of the rectangle.

V. Discussion

The situation which occurs when a portion of the boundary coincides with a characteristic of the reduced equation is very special. To indicate this, we consider a new domain R consisting of the interior of the parallelogram in Fig. 2. For later convenience we number the sides of the parallelogram as shown. Again the differential equation (2.1) is required to hold in R and the boundary values of the unknown function u are prescribed on all four sides. It will be assumed that $-\pi/2 < \varphi < \pi/2$.

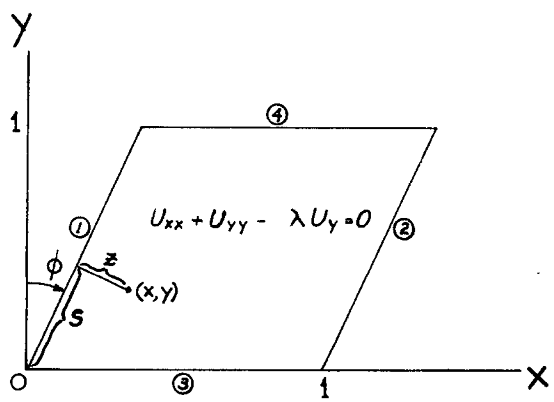

FIG. 2.

We found in Section III that the boundary layer along the "characteristic" boundary $x = 0$ in our example problem was governed by a partial differential equation (Eq. (3.4)). In order to analyze the boundary layer correction u_1 near side 1 of the parallelogram in the present case, it is convenient to introduce as local coordinates in the vicinity of side 1 the distance s along the side from the origin and the perpendicular distance z into R from the segment. These are given by

$$\begin{aligned} s &= x \sin\varphi + y \cos\varphi \\ z &= x \cos\varphi - y \sin\varphi. \end{aligned} \tag{5.1}$$

In terms of s and z the differential equation (2.1) becomes

$$u_{zz} + u_{ss} + \lambda \sin\varphi u_z - \lambda \cos\varphi u_s = 0. \tag{5.2}$$

If in this equation we assume $\varphi = 0$ and set

$$\xi = \lambda z \tag{5.3}$$

and then retain only the highest powers of λ, we find the following *ordinary* differential equation for the boundary layer correction $u_1(\xi, s)$.

$$\frac{\partial^2 u_1}{\partial \xi^2} + \sin\varphi \frac{\partial u_1}{\partial \xi} = 0. \tag{5.4}$$

If on the other hand $\varphi = 0$, the boundary layer scale is of the order of λ^{-1}, and u_1 satisfies the partial differential equation (3.4).

At side 2 of the parallelogram, the boundary layer scale is again λ^{-1} if $\varphi \neq 0$, and the differential equation for the boundary layer correction is

$$\frac{\partial^2 u_1}{\partial \bar{\xi}^2} - \sin\varphi \frac{\partial u_1}{\partial \bar{\xi}} = 0, \tag{5.5}$$

where $\bar{\xi}$ is the local boundary layer coordinate at side 2.

At the bottom edge (side 3) of the parallelogram we write

$$\bar{\eta} = \lambda y \tag{5.6}$$

and obtain the boundary layer equation

$$\frac{\partial^2 u_1}{\partial \bar{\eta}^2} - \frac{\partial u_1}{\partial \bar{\eta}} = 0. \tag{5.7}$$

At the top edge (side 4)

$$\bar{\eta} = \lambda(1 - y) \tag{5.8}$$

and

$$\frac{\partial^2 u_1}{\partial \eta^2} + \frac{\partial u_1}{\partial \eta} = 0. \tag{5.9}$$

According to (5.4) an exponentially decaying boundary layer on the length scale λ^{-1} can occur along side 1 if $0 < \varphi < \pi/2$ but cannot occur if

$-\pi/2 < \varphi < 0$. If $\varphi = 0$, there can be a boundary layer along side 1 of scale $\lambda^{-1/2}$, as in Section III. The situation reverses on side 2; there can be a boundary layer if $-\pi/2 < \varphi \leqslant 0$ (if $\varphi = 0$, again the scale is different and (3.4) replaces (5.5)) but not if $0 < \varphi < \pi/2$. From (5.7), side 3 can never support a boundary layer, but from (5.9) side 4 can always do so.[5] Figures 3(a) and 3(b) illustrate the location of the boundary layers in the "noncharacteristic" case and Figure 3(c) describes the situation in the "characteristic" boundary case.

In view of the possible location of the boundary layers, the *interior approximation* $u_0(x)$ must be made to satisfy the originally prescribed boundary

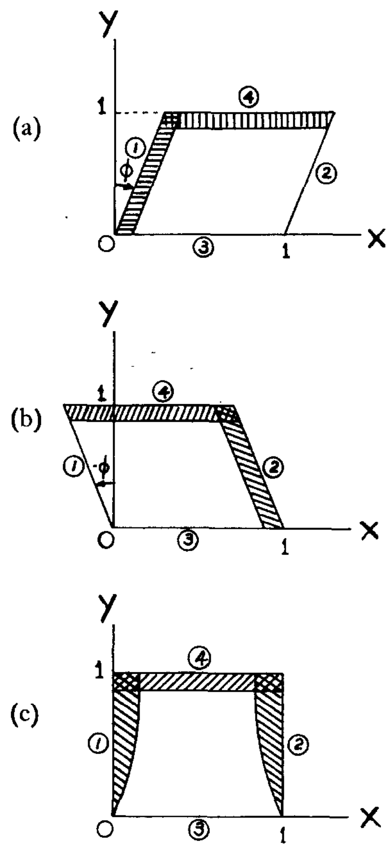

FIG. 3. Location of the boundary layers in noncharacteristic (a) and (b) and characteristic (c) boundary cases. Boundary layers indicated by shading.

[5] Some of the conclusions obtained here about the presence or absence of boundary layers along the various sides follow from the results of Levinson [5].

conditions along sides 2 and 3 in Fig. 3(a), along sides 1 and 3 in Fig. 3(b) and only along side 3 in Fig. 3(c).

We observe that when $\varphi \neq 0$, so that no portion of the boundary of the parallelogram coincides with a characteristic of the reduced equation, *all boundary layers are described by ordinary differential equations. This is in contrast with the case of the "characteristic boundary"* ($\varphi = 0$), *where the partial differential equation (3.4) is relevant.* In the characteristic case the corner layer was needed to provide the necessary additional boundary condition for the boundary layer partial differential equation. In the noncharacteristic case, there are still corner layers present, but they need not be analyzed unless information in the corners is specifically required.

Consider the general linear second order elliptic equation

$$u_{xx} + u_{yy} - \lambda(Au_x + Bu_y + Cu) = 0, \tag{5.10}$$

where A, B, and C are sufficiently smooth functions of x and y but do not depend on λ. The vector field $\mathbf{v}$ with x-component A and y-component B is tangent at each point to the characteristic curve through that point of the reduced equation associated with (5.10). Let S_0 be a smooth arc of the boundary S of the domain R in which (5.10) is required to hold. Let s denote arc length along S_0, and assume that the positive direction of s is such that the tangent vector to S_0 in the increasing s direction becomes the inward normal vector $\mathbf{n}$ to S_0 upon counterclockwise rotation through ninety degrees. Let z be distance into R from S_0 measured along $\mathbf{n}$. Suppose first that S_0 is nowhere tangent to a characteristic of the reduced equation, so that $\mathbf{v} \cdot \mathbf{n} \neq 0$. It can then be shown that the boundary layer variable for S_0 is

$$\xi = \lambda z$$

and that the boundary layer differential equation which follows from (5.10) is

$$u_{\xi\xi} - (\mathbf{v}_0 \cdot \mathbf{n})\, u_\xi = 0.$$

The subscript zero on $\mathbf{v}_0$ means that $\mathbf{v}$ is evaluated on S_0. The arc S_0 will therefore support a boundary layer only if $\mathbf{v}_0 \cdot \mathbf{n} < 0$, so that the vector field $\mathbf{v}_0$ points *out* of the domain R along S_0.

On the other hand suppose that S_0 everywhere coincides with a characteristic of the reduced equation. Then $\mathbf{v}_0 \cdot \mathbf{n} = 0$ along S_0 so that $\mathbf{v}_0$ is tangent to S_0, and the boundary layer variable for S_0 turns out to be

$$\xi = \lambda^{1/2} z.$$

The boundary layer differential equation which results is

$$u_{\xi\xi} \pm (A_0^2 + B_0^2)^{1/2}\, u_s - C_0 u = 0.$$

The subscript zero again indicates evaluation on S_0, and it is assumed that $A_0^2 + B_0^2 \neq 0$.

Thus the boundary layer along S_0 in described by a (parabolic) partial differential equation (rather than an ordinary differential equation) if and only if S_0 coincides with a characteristic of the reduced equation.

VI. Internal Layers

There is another aspect to the role of the characteristics of the reduced equation in problems of the type we are considering. The interior approximation $u_0(x)$ must be determined as a solution of the reduced equation satisfying the original boundary conditions along those parts of the boundary of the domain which cannot support boundary layers. Under certain conditions this may introduce discontinuities into the interior approximation or its derivatives. From the theory of partial differential equations, it is clear that such discontinuities can only occur across characteristics of the reduced equation (the lines $x =$ constant in the present problem). The original differential equation (2.1), however, is elliptic and its solutions must in fact be smooth throughout the domain considered. It follows that the interior approximation u_0 does not properly describe the exact solution in the vicinity of a discontinuitycarrying characteristic, and the higher order terms in (2.1) must become important there. This indicates that along any such characteristic there must be "*internal layers*" (analogous to boundary layers) which serve to smooth out the discontinuity. Since these internal layers occur along characteristics, the relevant corrections (one on each side of the line of discontinuity) to the interior approximation must again satisfy the heat conduction equation (3.4). The appropriate boundary conditions follow from the requirements of continuity across the characteristic of the exact solution and its first derivative, and from corner layer considerations of the type discussed in Section III. With this formulation the internal layers can be easily calculated.

One way in which discontinuities can be introduced into the interior approximation u_0 is of course directly by way of discontinuous boundary data along that portion of the boundary on which $u_0(x)$ must satisfy the original boundary condition. A somewhat more interesting source of discontinuities is indicated in Fig. 4. Suppose the Dirichlet problem for Eq. (2.1) is again posed, but now in the domain R of Fig. 4. and suppose that the boundary data are smooth functions of position. The discussion at the beginning of this section would suggest that there can be no boundary layers along the arc $ab''d$ or bc, but that they can occur along ab', $b'c$ and bd. The interior approximation in Region I would then be determined by the boundary values given on ad; however, the interior solution in Region II is determined by boundary data on bc. Since in the present problem the interior approximation u_0 depends only on x, it is clear that there will be a discontinuity in $u_0(x)$ across the line bb' unless the boundary values at b and b'' happen to be the same. Therefore in general internal layers on either side of the line segment bb' would be required.

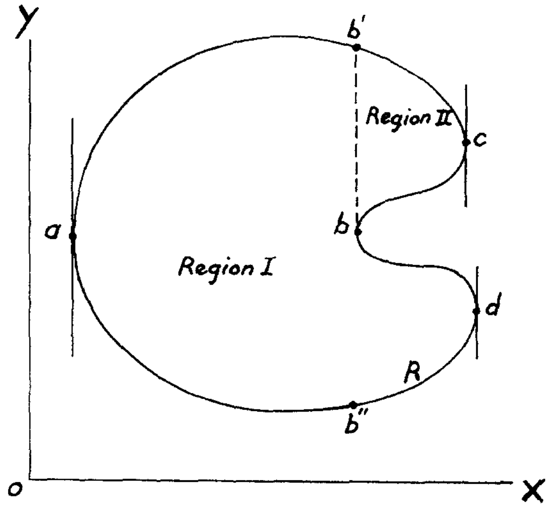

FIG. 4. Domain illustrating discontinuities in the interior approximation.

Internal boundary layer phenomena are also discussed in [2].

The location of boundary- and internal-layers as described in this and the preceding sections becomes particularly clear when viewed in terms of the conduction-convection problem for the moving plate. The situations depicted in Fig. 3 and 4 can also be interpreted as the exposed portions of thin heat-conducting plates moving with a large velocity proportional to λ in the positive y direction and surrounded by suitable heat reservoirs. The temperature in the interior of the plates is determined entirely by convection (see Eq. (2.3)). In Fig. 3a, this has the effect of "carrying" into the interior parallel to the y-axis the temperatures along sides 2 and 3. Since the temperatures prescribed on sides 1 and 4 may differ from those arising due to convection large temperature gradients must develop in narrow layers near these sides. In these layers diffusion effects become of importance comparable to those due to convection. This gives rise to the required boundary corrections. Corresponding remarks apply to Figs. 3b and 3c. In Fig. 4 the differing temperatures resulting in Regions I and II due to convection from arcs $ab'd$ and bc, respectively, will in general give rise to large temperature gradients in the x-direction across bb', and the diffusion process will again introduce the required smoothing.

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